2021 TJPhO Solutions

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1 Background Problems

Electric and Magnetic Fields

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a) Two charges $-q$ and $2q$ are at $x = -a$ and $x = a$ respectively. A test charge is placed at $x = d$. Find d if the force on the test charge is zero.

b) Suppose an infinite straight wire has constant current I. Taking C to be a circular loop with radius r and appealing to symmetry, use Ampere's Law to determine the magnitude of the magnetic field at a distance r from the wire.

c) An electric field E and magnetic field B are directed along the $+z$-axis. A mass m with charge q starts at the origin with velocity v along the $+x$-axis. Find the position of the particle as a function of time and describe the motion.

Solution.

a) The three cases of d are $d < -a$, $-a < d < a$, and $d > a$. It can be dynamically inferred that the cases $d < -a$ and $d > a$ cannot hold true as the net force upon d cannot be 0 for these cases (the forces at $-a$ and a do not cancel). Thus, $-a < d < a$ is the only acceptable case for d. Since we want the net force at d to be 0, we can say that the sum of the Lorentz forces at $-a$ and a must be 0. The charges at $-a$ and a have velocity 0, and the distances from $-a$ to d and d to a are $d+a$ and $a-d$, respectively. From this, we get the equation $-\frac{1}{(d+a)^2} + \frac{2}{(a-d)^2} = 0$ by the Lorentz Force Equation and Coulomb's Law. This leads to the answer $d = -3a \pm 2a\sqrt{2}$. To check for extraneous solutions, we can check the base case $a = 1$. This leads us to the values $-3 \pm 2\sqrt{2}$ for d. The former holds true for the case $-a < d < a$ and correlates to $d = -3a + 2a\sqrt{2}$, while the latter does not hold true for the chosen case. Therefore, $d = -3a + 2a\sqrt{2}$.

b) $\oint_c \mathbf{B} \cdot d\mathbf{r} = \mu_0 I$ is the given equation for Ampere's Law. In order to find the magnitude of the magnetic field, we must derive a new equation (note that the strength of the magnetic field is the magnitude). The newly derived equation will be in terms of the variable $\mathbf{B}$. Deriving the new equation will only have 2 steps; first we must convert $d\mathbf{r}$ into $2\pi r$. After this we have the equation; $\oint_c \mathbf{B} \cdot 2\pi\mathbf{r} = \mu_0 I$. The next step would be to divide the left hand side by $2\pi\mathbf{r}$, which would in turn cause the right hand side to be divided by $2\pi\mathbf{r}$. The newly derived equation would be $\mathbf{B} = \frac{\mu_0 I}{2\pi\mathbf{r}}$. Since the given values of r and I are just r and I, respectively, the final magnitude of the magnetic field at a distance r from the wire is simply $\frac{\mu_0 I}{2\pi\mathbf{r}}$ T.

c) From Coulomb's Law, it can be derived that m's electric field is $E = \frac{q\hat{r}}{4\pi\epsilon_0 r^2}$. By Ampere's Law, it can be determined that m's magnetic field is $B = \frac{\mu_0 I}{2\pi r}$. Now we can use a rearrangement of the Lorentz Force Equation to determine the velocity, which leads us to $v = \frac{2\pi rF}{q\mu_0 I} - \frac{\hat{r}}{2\mu_0 Ir\epsilon_0}$. Since position is the time-integral of velocity, we can model m's position as the position function $x(t) = \int vdt = \int(\frac{2\pi rF}{q\mu_0 I} - \frac{\hat{r}}{2\mu_0 Ir\epsilon_0})dt = \frac{2\pi rFt}{q\mu_0 I} - \frac{\hat{r}t}{2\mu_0 Ir\epsilon_0}$. The constant of integration is disregarded since the initial position of m is $x = 0$ (in other words, the constant of integration is 0, which does not have to be written). Since m's velocity is constant, it can be kinematically inferred that the acceleration of m is 0. Therefore, the mass m with charge q moves at velocity $v = \frac{2\pi rF}{q\mu_0 I} - \frac{\hat{r}}{2\mu_0 Ir\epsilon_0}$ and acceleration 0, and m's position can be modeled as $x(t) = \frac{2\pi rFt}{q\mu_0 I} - \frac{\hat{r}t}{2\mu_0 Ir\epsilon_0}$.

Conductors

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1. Find the resistance of a conductor with cross-sectional area A, length L, and conductivity σ in terms of the given variables.

2. After a long time, the electrons in a current will move with constant drift velocity v_d. Find v_d for a conductor placed in a uniform electric field E.

3. Suppose the conductor has electron density n and conductivity σ. Show the average time between electron collisions is $\tau = \frac{m\sigma}{e^2 n}$.

Solution.

1. From Ohm's Law, we can derive that $R = \frac{V}{I}$. By the Electric Potential Difference formula, we can easily integrate to find V as such: $V = -\int_L^0 E d\ell = EL$. By Ohm's Law, we can also derive that $I = AJ = A\sigma E$. This leads us to $R = \frac{V}{I} = \frac{EL}{A\sigma E} = \frac{L}{A\sigma}$. Therefore, the resistance of the conductor is $R = \frac{L}{A\sigma}$.

2. Let $e = q$, where q was originally the variable for the electron charge. At the start, we can use $F = ma$. We switch around the equation to $a = \frac{F}{m}$. This gives us the acceleration and we can use this in terms of the electric field, which gives us $a = \frac{eE}{m}$. Within the conductor, these electrons collide very often, and we can use τ to define the time between collisions. Thus, we can integrate both sides with respect to time in order to find v_d as velocity is the time-integral of acceleration. This leaves us with $v_d = \frac{e\mathbf{E}\tau}{m}$. Therefore, $v_d = \frac{eE\tau}{m}$ for a conductor placed in a uniform electric field E.

3. Suppose that e is the charge of electrons. To begin, we use a rearrangement of the drift velocity formula: $\tau = \frac{v_d m}{eE}$. Since $E = \frac{J}{\sigma}$ by Ohm's Law and E is uniform, it can be inferred that J and σ are equivalent. Since E is uniform, it can also be inferred that $E = \frac{e}{4\pi\epsilon_0}$ by Coulomb's Law (r must be 1 since E is uniform and $\hat{r}$ is negligible). By relating the two established values for E, we can get the following identities: $J = e$, $\sigma = 4\pi\epsilon_0$, $E = 1$, $J = e = \sigma$. Now by substituting these identities into the τ equation, we get $\tau = \frac{v_d m}{eE} = \frac{v_d m\sigma}{eJ} = \frac{v_d m\sigma}{e^2}$. Now we can assume $n = v_d^{-1} = \frac{m}{eE\tau}$. This can be directly proved by substituting the assumed value of n into the τ equation: $\tau = \frac{m\sigma}{e^2 n} = \frac{m\sigma}{e^2 \frac{m}{eE\tau}} = \frac{m\sigma E\tau}{em}$. Since E is 1 and $e = \sigma$, $\tau = \frac{m\sigma E\tau}{em} = \tau$, which is an identity that implies the assumed value of n holds true. Therefore, $\tau = \frac{v_d m}{eE} = \frac{v_d m\sigma}{eJ} = \frac{v_d m\sigma}{e^2} = \frac{m\sigma}{e^2 n}$.

Circuits

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1. Joule's first law states the resistor will dissipate power $P = IV = I^2R$ in the form of heat. By conversation of energy, an equal power must be delivered to the circuit. Where does this power come from?

Solution.

1. Within a resistor, electrons move around and collide. These electrons create small amounts of kinetic energy which then are dissipated into heat energy.

2 BCS Theory

Quantum Statistics

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1. Using the spin-statistics theorem, show that bosons have integral spin while fermions have half-odd-integral spin.
2. Justify the Pauli exclusion principle, which states that two fermions can not occupy the same quantum state.

Solution.

1. Let x map to r_1, r_2 and $-x$ to r_2, r_1. $\psi(x) = \psi(-x)$ for bosons from the spin-statistics theorem. $\psi(x) = -\psi(-x) = -\psi(x)$ for fermions from the spin-statistics theorem. This implies that the wavefunction is symmetric for bosons and anti-symmetric for fermions. In quantum mechanics, symmetry implies an integer spin. Therefore, bosons must have an integral spin because of their even wavefunction, and fermions must have a non integral odd spin as implied by their odd wavefunction.
2. The wavefunction for a fermion is antisymmetric. Hence, under an exchange of the two position vectors of the fermion wavefunction, one form is positive while the other is negative. Therefore, if one side of the wavefunction for a fermion produces positive spin, it implies that its exchange leads to a negative spin. Their different spins imply a difference in quantum states. Thus, the antisymmetric wavefunction for a fermion leads to positive spin on one side and negative spin under a exchange symmetry, implying a difference in quantum states.